Associated primes

Motivation:

Let n be an integer.

We can write its unique prime factorization as

$$n = \pm p_1^{d_1} \cdots p_t^{d_t}.$$

In fact, in $\mathbb{Z}$, $(n) = (p_1^{d_1}) \cap \dots \cap (p_t^{d_t})$.

We will see that the "associated primes" of (n) are the (p_i) and the primary components of (n) are the $(p_i^{d_i})$.

We will use these concepts to generalize the unique factorization of integers to arbitrary rings.

Geometric motivation:

Let $R = k[x_1, \dots, x_n]$, and $I \subseteq R$ an ideal.

Def: The closed set $V(I)$ is <u>reducible</u> if it can be written $V(I) = V(I') \cup V(I'')$ where $V(I)$ is not equal to $V(I')$ or $V(I'')$.

Otherwise $V(I)$ is <u>irreducible</u>.

Claim: $V(I)$ is irreducible $\iff \sqrt{I}$ is prime.

Pf: If $\sqrt{I}$ is prime, then if $V(I) = V(\sqrt{I}) = V(I') \cup V(I'')$ and $\sqrt{I} \in V(I')$, then $V(I') \supseteq V(\sqrt{I})$, so they're equal.

If $\sqrt{I}$ isn't prime, $fg \in \sqrt{I}$ for $f, g \notin \sqrt{I}$.

So for $P \in V(\sqrt{I})$, $f \in P$ or $g \in P$.

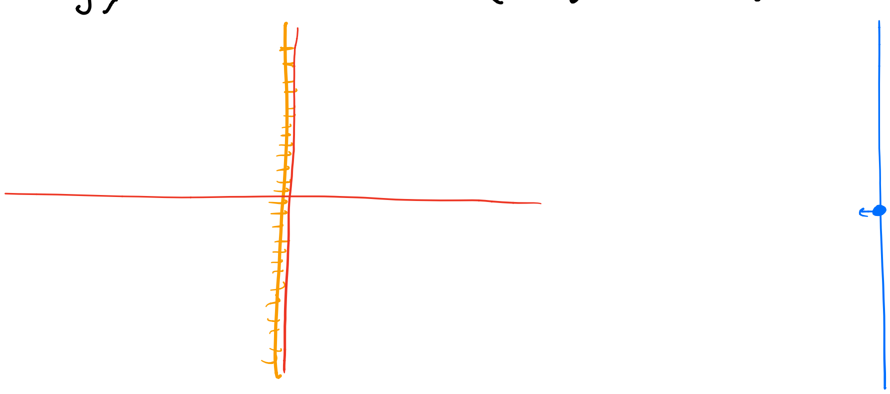
$$\Rightarrow V(\sqrt{I}) = V(\sqrt{I}, f) \cup V(\sqrt{I}, g)$$

Neither of which is equal to $V(\sqrt{I})$ since $f, g \notin \sqrt{I} = \bigcap_{\substack{P \supset I \\ \text{prime}}} P.$ $\square$

We'll see that $\sqrt{I}$ can be written in a unique minimal way as a finite intersection of primes. This is the "primary decomposition" of $\sqrt{I}$ and corresponds to writing $V(I)$ in the unique minimal way as the union of irreducible closed sets.

Ex: Define $I := (x^2, xy) \subseteq k[x,y]$.

Geometrically, this is $V(x^2) \cap V(xy)$



which is, geometrically, roughly the line $x=0$ w/ additional "scheme" structure (i.e. a tangent direction) at the origin.

We will see, purely algebraically, that the associated primes are (x) and (x,y).

However, we can write $I = (x) \cap \underset{\substack{\uparrow \\ \text{radical} = (x,y)}}{(x^2, y)}$ or $I = (x) \cap \underset{\substack{\uparrow \\ \text{radical} = (x,y)}}{(x^2, xy, y^2)}$

So the description as the intersection of ideals whose radicals are the associated primes is not unique.

More precisely, let R be a ring and M an R-module.

Def: A prime P of R is associated to M if there is some $x \in M$ s.t. $P = \operatorname{ann}(x) = \{r \in R \mid rx = 0\}$.

The set of all primes associated to M is denoted $\operatorname{Ass}_R M$, or just $\operatorname{Ass} M$ if the ring is clear.

(Sometimes the associated primes of R/I over R are just called the associated primes of I.)

Remark: If $P \in \operatorname{Ass} M$, then $P = \operatorname{ann}(x)$, so

$R \xrightarrow{\cdot x} M$ has kernel P, so $R/P \cong$ a submodule of M.

Conversely, if P is some prime ideal s.t. $R/P \hookrightarrow M$ as modules, then P is the annihilator of the image of 1.

That is:

P is an associated prime of M $\iff$ R/P is isomorphic to a submodule of M.

Now we state some important results about associated primes

Theorem: Let R be a Noetherian ring and $M \neq 0$ a finitely generated R-module. Then

a.) $\text{Ass}\, M$ is finite and nonempty, each containing $\text{ann}(M)$. It includes all primes minimal among those containing $\text{ann}\, M$.

b.) $\bigcup_{P \in \text{Ass} M} P = \{\text{zerodivisors on } M\} \cup \{0\}$

c.) $\text{Ass}\, M$ commutes w/ localization. i.e. if $U \subseteq R$ is multiplicatively closed, then

$$\text{Ass}_{R[U^{-1}]} M[U^{-1}] = \{PR[U^{-1}] \mid P \in \text{Ass}\, M \text{ and } P \cap U = \emptyset\}.$$

We'll prove this in the next section after a few lemmas.

Remark: Why can we find primes minimal over an ideal?

Let $\{Q_i\}$ be a chain of prime ideals containing I.

Then if $ab \in \bigcap Q_i$, a or b is in all Q_i, so $\bigcap Q_i$ is prime.

That is, every chain has a lower bound, so Zorn's Lemma implies that there exist minimal primes over I.

(Note that this holds for even non-Noetherian rings!).

Def: The primes in $\operatorname{Ass} M$ that are not minimal are called <u>embedded</u> primes of M.

If $M = R/I$, then if P is an embedded prime in R, $V(P)$ is called an <u>embedded component</u> of $\operatorname{Spec}(R/I)$.

If P is a minimal associated prime, $V(P)$ is an <u>isolated component</u> of $\operatorname{Spec}(R/I)$.

Ex: Let's go back to the example of $I = (x^2, xy) \subseteq R$.

What is $\operatorname{Ass}(R/I)$? The only nonzero elements annihilated are multiples of x or y.

$\operatorname{ann}(x) = (x,y)$ and $\operatorname{ann}(y) = (x)$. $\Rightarrow \operatorname{Ass}(R/I) = \{(x), (x,y)\}$

embedded prime